Math 254B Lecture 23 Notes

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1 Iterated Function Systems

1.1 Similitudes and iterated function systems

Definition 1.1. A **similitude** in $\mathbb{R}^d$ is a map $\Phi : \mathbb{R}^d \to \mathbb{R}^d$ with some $r > 0$ such that $|\Phi x - \Phi y| = r|x - y|$ for all x, y. A similitude is **contracting** if $r < 1$.

Similitudes take the form $\Phi(x) = rUx + a$, where U is orthogonal.

Definition 1.2. An **iterated function system** (**IFS**) is a finite sequence $\Phi = (\Phi_i)_{i=1}^k$ of contracting similitudes.

Remark 1.1. We can construct a new IFS by composition: Given $w = (w_1, \dots, w_n) \in [k]^n$, Let $\Phi_w = \phi_{w_1} \circ \cdots \circ \phi_{w_n}$, This gives a new IFS $(\Phi_w)_{w \in [k]^n}$.

1.2 The Hausdorff metric and existence of attractors

Our usual examples, Cantor sets, the von Koch curve, and the Sierpiński carpet, can be constructed from iterated function systems.

Theorem 1.1. *For every IFS $\Phi = (\Phi_i)_{i=1}^k$, there exists a unique nonempty compact $K \subseteq \mathbb{R}^d$ (called the **attractor** of Φ) such that $K = \bigcup_{i=1}^k \Phi_i[K]$ Moreover, if L is nonempty, compact, and $\Phi_i[L] \subseteq L$ for all I, then*

$$\bigcup_{w \in [k]^n} \Phi_w[L]$$

decreases to K.

The idea to prove this is to use the Banach contraction mapping theorem. So we should define $\mathcal{S} = \{L \subseteq \mathbb{R}^d : L \neq 0, L \text{ compact}\}$.

Definition 1.3. Define the **Hausdorff metric** on $\mathcal{S}$ as

$$\rho_H(K, L) := \inf \left\{ \delta > 0 : K_\delta = \bigcup_{x \in K} B_\delta(x) \supseteq L, L_\delta \supseteq K \right\}.$$

Lemma 1.1. *ρ_H is a metric on $\mathcal{S}$.*

Proof. For the triangle inequality, use the following: if $L \subseteq K_\delta$ and $K \subseteq M_\varepsilon$, then $M_{\varepsilon+\delta} \supseteq K_\delta \supseteq L$. $\square$

Lemma 1.2. *$(\mathcal{S}, \rho_H)$ is complete.*

Proof. Suppose $(K_n)_n$ is Cauchy. Then for every $\varepsilon > 0$, there is a $n_0 \geq 1$ such that for all $n, m \geq n_0$, we have $J_n \subseteq (K_m)_\varepsilon$ and $K_m \subseteq (K_n)_\varepsilon$. From this, conclude that for all $x \in \mathbb{R}^d$, either:

- there is a $\delta > 0$ such that $B_\delta(x) \cap K_n \neq \varnothing$ for all sufficiently large n
- for all $\delta > 0$, we have $B_\delta(x) \cap K_n \neq \varnothing$ for all sufficiently large n.

The set of points for which the latter condition holds are the limit set. This set is bounded, and the complement, the set obeying the former condition, is a union of open sets. Now use compactness. $\square$

Lemma 1.3. *Given $\Phi = (\Phi_i)_{i=1}^k$, define $\tilde{\Phi} : \mathcal{S} \to \mathcal{S}$ by $\tilde{\Phi}(K) = \bigcup_i \Phi_i(K)$. Then $\tilde{\Phi}$ contracts ρ_H.*

Proof. If $K \subseteq L_\varepsilon$, then $\Phi_i[K] \subseteq (\Phi_i[L])_{r^*\varepsilon}$, where $r^* = \max_i r_i < 1$. Then $\tilde{\Phi}^i[K] \subseteq (\tilde{\Phi}[L])_{r^*\varepsilon}$. so $\rho_H(\tilde{\Phi}[K], \tilde{\Phi}[L]) \leq r_* \rho_H(K, L)$. $\square$

Now we can prove the existence theorem.

Proof. Attraction is if and only if $K = \tilde{\Phi}(K)$. Now use the Banach contraction mapping theorem. For any other $L \in \mathcal{S}$, we get $\rho_H(K, \tilde{\Phi}^t(L)) \xrightarrow{t\to\infty} 0$. If $L \supseteq \Phi_i[L]$ for all i, then $L \supseteq \tilde{\Phi}[L] \supseteq \cdots \supseteq \tilde{\Phi}^t[L]$ for all t. So $\lim_t \tilde{\Phi}^t[L] = \bigcap_t \tilde{\Phi}^t[L]$. $\square$

1.3 Coding maps

Consider again an IFS $\Phi = (\Phi_i)_{i=1}^k$. Given $w = (w_1, \ldots, w_n) \in [k]^n$, then we define $\Phi_w := \Phi_{w_1} \circ \cdots \circ \Phi_{w_n}$.

Lemma 1.4. *If $\omega \in [k]^{\mathbb{N}}$ and $x \in \mathbb{R}^d$, then $\Phi_{\omega|_1^n}(x)$ converges to a limit $\pi(\omega)$, independent of x.*

Proof. Let $r^* = \max_i r_i < 1$. If D is a big enough closed ball, then $\Phi_D \subseteq D$ for all i. Consider $x, y \in \mathbb{R}^n$. For any $w \in [k]^n$, we get $|\Phi_w(x) - \Phi_w(y)| \leq (r^*)^n |x - y|$. On the other hand, if $x \in D$, then any $\Phi_v(x)$ is still in D. So

$$|\Phi_{\omega|_1^n}(x) - \underbrace{\Phi_{\omega|_1^m}(x)}_{=\Phi_{\omega|_1^n} \circ \Phi_{\omega|_{n+1}^m}(x)}| \leq (r^*)^n(\operatorname{diam}(D)).$$

So this is a Cauchy sequence and hence converges. We also get that the limit is independent of x. $\square$

Here is another consequence: If $\omega, \omega' \in [k]^{\mathbb{N}}$ with $\omega|_1^n = \omega'|_1^n$, then

$$|\pi(\omega) - \pi(\omega'| = \lim_t |\Phi_{\omega|_1^t}(x) - \Phi_{\omega'|_1^t}(x)| \leq (r^*)^n \operatorname{diam}(D).$$

Definition 1.4. We refer to $\pi : [k]^{\mathbb{N}} \to \mathbb{R}^d$ as the **coding map**.

On $[k]^{\mathbb{N}}$, define $\Psi_i : [k]^{\mathbb{N}} \to [k]^{\mathbb{N}}$ sending $\omega \mapsto i\omega$ (append i to the beginning of the infinite word). This is a symbolic version of Φ_i because

$$\pi \circ \Psi_i(\omega) = \pi(i\omega) = \lim_t \Phi_i \circ \Phi_{\omega_1} \circ \cdots \circ \Phi_{\omega_t}(x) = \Phi_i(\pi(\omega)).$$

$$\begin{array}{ccc} [k]^{\mathbb{N}} & \xrightarrow{\Psi_i} & [k]^{\mathbb{N}} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ K & \xrightarrow{\Phi_i} & K \end{array}$$

You can think of $\pi(\omega)$ as sending an "address" to its corresponding point.

Remark 1.2. π need not be injective; i.e. a point can have multiple addresses.

1.4 Conditions for iterated function systems

Definition 1.5. We say that Φ satisfies the **strong separation condition** (**SSC**) if for $i \neq j$, $\Phi_i[K] \cap \Phi_j[K] = \varnothing$.

Φ satisfies the SSC iff π is injective.

Example 1.1. The IFSs generating Cantor set and Cantor dust satisfy the SSC.

Example 1.2. The IFSs generating the Sierpiński carpet and the von Koch curve do not satisfy the SSC.

Definition 1.6. We say that Φ satisfies the **open set condition** (**OSC**) if for there is a nonempty open set such that $\Phi_i[U] \subseteq U$ for all i and such that when $i \neq j$, then $\Phi_i[U] \cap \Phi_j[U] = \varnothing$.

U may not contain K.

Example 1.3. The IFSs generating the Sierpiński carpet and the von Koch curve satisfy the OSC.